

## **SINGLE VERSUS MULTIPLE DISCOUNT RATES IN INVESTMENT THEORY**

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### **Abstract**

The conventional procedure to estimate the rate of return on an investment project appears to contain significant errors on three counts. First, it uses a single discount rate as an estimator for all the periodic changes in the market rates that take place during the life of a project which may either overestimate or underestimate the net rate of return. The use of a single discount rate to present value the future cash flows will result in two types of errors: (a) errors committed during the estimation of individual one-period rates, and (b) errors committed during the estimation of a single discount rate as an estimator of the individual one-period rates. Second, it overestimates (under estimates) the rate of return by excluding the increase (decrease) in the opportunity cost of the various segments of the price paid from the time the cash inflows accrue until the termination or the sale of the project. If the opportunity costs increase (decrease) during the investment period, then the actual rate of return will be lower (higher). It is inconsistent in including only the time value of the cash inflows while excluding the time value of their corresponding prices. Third, it overemphasizes the geometric-mean-based procedure to calculate the rate of return, while ignoring other equally excellent procedures such as harmonic-mean-based and arithmetic-mean-based.

Recognizing that conventional methods of calculating the rate of return on various investment projects often yield significant errors, this paper introduces a new procedure to correct the inconsistency in the estimation of the rate of return by explicitly recognizing the time value of the various segments of the project's price (cost). The choice of a particular mean-based (harmonic, geometric, or arithmetic) technique to calculate the rate of return can be linked to the risk preference of the investors. The results of this paper indicate that under variable rate conditions a harmonic-mean-based rate of return will always be lower than a geometric-mean-based rate of return which, in turn, will always be lower than an arithmetic-mean-based rate of return. The differences in the values of these three mean-based rates of return are very small. With regard to the adoption of an investment project (or to the purchase of an asset), therefore, the cautious risk-averse investor is more likely to make a decision based upon the harmonic-mean based rate of return; the flamboyant risk preferrer investor is more likely to make a decision based upon the arithmetic-mean-based rate of return; and a risk-neutral investor is more likely to make a decision based upon the geometric-mean-based rate of return. All of the three measures to calculate the rate of return are equally effective in appropriate contexts. This paper uses multiple single-period rates as opposed to a single discount rate to evaluate a project's net worth.

### **INTRODUCTION**

The literature, generally, uses a single discount rate as an estimator for all the periodic changes in the market rates that take place during the life of a project which may either overestimate or underestimate the net rate of return on a project. The use of a single discount rate to present value the future cash flows will result in two types of errors: (a) errors committed during the estimation of individual one-period rates, and (b) errors committed during the estimation of a single discount rate as an estimator of the individual one-period rates. The opportunity

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cost of the price paid for the project rises when market rates, on the average, rise during the life of a project and vice-versa. Only if the rates remain constant or if they change in a neutralizing fashion, the present value of the future net cash flows using any method (a single discount rate or multiple one-period rates) will yield identical results. The Expectations School of the term structure theory claims that the long-term rates are simply geometric averages of the short-term rates. Based upon this, one can infer, then, that an average of one-period rates can be used as a single discount rate to obtain the present worth of the future cash inflows on a project. It is demonstrated in Section 1 that the utilization of a single discount rate, obtained by using any type of estimation technique (OLS, GLS, 2SLS, 3SLS, k-class estimator) or by taking an average (geometric, harmonic, or arithmetic) of one-period rates, will produce an inaccurate estimate in the present valuation process. The use of a single discount rate as opposed to multiple one-period rates in the "present valuation" of a project may either overestimate or underestimate the net rate of return.

The principal reason for the continued success and existence of the discipline of finance is that it introduces various investment strategies, which, if wisely followed can, to a reasonable extent, guarantee the highest rate of return to investors. For the better or the otherwise, the maximization of the monetary rate of return seems to be the primary motivating force behind all the human activities today. Therefore, the way the monetary rate of return is calculated is of great interest to investors. Various methods can be used to calculate the rate of return. The most commonly used method in the literature to calculate the rate of return on investment projects is geometric-mean based. Utilizing the methodologies to calculate the geometric-mean, harmonic-mean, and arithmetic-mean, this paper proposes an adjustment for the continuous changes in the opportunity costs of the investment projects during the investment horizon.

Furthermore, generally, the literature uses a single discount rate to estimate the present worth of a project's future net cash flows. Since the market rates change constantly, it would only seem appropriate to adopt multiple single-period rates in the present valuation of projects and in their ranking process, rather than a single discount rate which is most commonly used in the literature. While calculating the rates of return on these projects, various types of mean-based evaluation techniques can be used: harmonic mean, geometric mean, and arithmetic mean. Under variable rate conditions, the harmonic-mean-based rate of return is always smaller than the geometric-mean-based rate of return which, in turn, is always smaller than the arithmetic-mean-based rate of return. These three methods can be complementary to each other. Investors with varying degree of risk preference will assign different values for the probabilities of default and recovery rates. A risk-averse investor tends to undervalue the rate of return on a project than a risk-neutral investor, while a risk-preferrer investor tends to overvalue it. Suppose the probabilities of default and recovery rates are not adjusted for various degrees of risk preferences of investors. Since the values of the rates of return based on harmonic-mean, geometric-mean, and arithmetic-mean techniques always lie in ascending order under variable rate conditions, different investors are likely to use different evaluation techniques to calculate a project's net worth: a risk-averse investor is likely to choose the harmonic-mean-based rate of return evaluation technique, a risk-neutral investor is likely to choose the geometric-mean-based rate of return evaluation technique, while a risk-neutral investor is likely to choose an arithmetic-mean-based rate of return evaluation technique.

The paper is organized as follows. Section 1 demonstrates that the use of one-period multiple discount rates would yield a better estimates than the use of a single discount rate in present valuation models. Section 2 explains the intuitive reasoning, supported by a numerical example, that lies behind the analysis of the subsequent sections. Section 3 introduces a new approach (geometric-mean-based, harmonic-mean-based, and arithmetic-mean-based) for project evaluation by adopting a piecemeal technique to incorporate the time value of the project's price (cost), as the time period in the investment horizon elapses. The rates of return based upon this new approach will be lower (higher) than the rates of return based upon the traditional approach if, on the average, the opportunity costs rise (fall). The opportunity-cost-adjusted rates of return are compared with the non-adjusted rates of return in section 4. The geometric-mean-based approach may not be globally superior. The issue of risk preference and the choice of a mean-based rate of return calculative technique has been discussed in section 5. Section 6 raises the issue of whether one mean-based rate of return could be derived as a special case of the other two. Section 7 provides some concluding remarks.

### MULTIPLE SINGLE-PERIOD RATES VERSUS A SINGLE DISCOUNT RATE

In this section it will be shown that the utilization of a single discount rate, obtained by using any type of estimation technique (OLS, GLS, 2SLS, 3SLS, k-class estimator) or by taking an average (geometric, harmonic, or arithmetic) of one-period rates, will produce an inaccurate estimate in the present valuation process. The use of a single discount rate as opposed to multiple one-period rates in the "present valuation" of a project may either overestimate or underestimate the net rate of return.

In the following analysis, it is assumed, for simplicity, that investors adopt a one-period reinvestment strategy. That is, the periodic cash inflows are reinvested in single-period investment projects and are continuously renewed (reinvested) each period until the end of the investment horizon. The same logic applies to other investment strategies such as two-period strategy, three-period strategy, four-period strategy, etc. The corresponding rates can be referred to as the investors' investment strategy matching-rates. For example, if an investor chooses a k-period reinvestment strategy, then the applicable rate is the k-period rate ( $k = 1, 2, 3, \dots, n$ ). To present value the future cash flows, investor strategy-matching-rates should be used. Thus for present valuation, one-period multiple rates should be used for investors that adopt a one-period reinvestment strategy, two-period multiple rates should be used for investors that adopt a two-period reinvestment strategy, three-period multiple rates should be used for investors that adopt a three-period reinvestment strategy, and n-period rates should be used for investors that adopt an n-period reinvestment strategy.

The present value of future cash flows using a single (s) discount rate,  $PV_s$ , is given by:

$$PV_s = \frac{C_1}{1+Y} + \frac{C_2}{(1+Y)^2} + \frac{C_3}{(1+Y)^3} + \dots + \frac{C_{n-1}}{(1+Y)^{n-1}} + \frac{C_n + F}{(1+Y)^n}$$

Let  $PV_m$  represent the present value of future cash flows using one-period multiple (m) rates as discounting factors such that:

$$PV_m = \frac{C_1}{1+Y_1} + \frac{C_2}{(1+Y_1)(1+Y_2)} + \frac{C_3}{(1+Y_1)(1+Y_2)(1+Y_3)} + \dots$$

$$+ \frac{C_{n-1}}{(1+Y_1)(1+Y_2)(1+Y_3)\dots(1+Y_{n-1})} + \frac{C_n + F}{(1+Y_1)(1+Y_2)(1+Y_3)\dots(1+Y_n)}$$

If Y is an unbiased estimator of the one-period multiple rates ( $Y_1, Y_2, Y_3, \dots, Y_n$ ), then:

$$PV_s = PV_m$$

such that:

$$\frac{C_1}{1+Y} + \frac{C_2}{(1+Y)^2} + \frac{C_3}{(1+Y)^3} + \dots + \frac{C_{n-1}}{(1+Y)^{n-1}} + \frac{C_n + F}{(1+Y)^n}$$

$$= \frac{C_1}{1+Y_1} + \frac{C_2}{(1+Y_1)(1+Y_2)} + \frac{C_3}{(1+Y_1)(1+Y_2)(1+Y_3)} + \dots$$

$$+ \frac{C_{n-1}}{(1+Y_1)(1+Y_2)(1+Y_3)\dots(1+Y_{n-1})} + \frac{C_n + F}{(1+Y_1)(1+Y_2)(1+Y_3)\dots(1+Y_n)}$$

However, the above equation is a polynomial of degree  $n$  and therefore, a solution for  $Y$  in terms of the one-period multiple rates ( $Y_1, Y_2, Y_3, \dots, Y_n$ ), periodic cash flows ( $C_1, C_2, C_3, \dots, C_n$ ), and  $F$  is not possible.  $Y$  can only be approximated either through a truncating system, or by taking an average (geometric, harmonic, arithmetic, or contraharmonic) of the one-period multiple rates. The investors have to form their investment strategies at the beginning of the investment horizon, where only  $Y_1$  is known with certainty. The other one-period rates (i.e.,  $Y_2, Y_3, Y_4, \dots, Y_n$ ) are unknowns. These rates can be estimated using an estimation technique (OLS, GLS, 2SLS, 3SLS, k-class estimator) with a reasonable degree of confidence. The use of a single discount rate to present value the future cash flows will, thus result in two types of errors:

1. **Type I Error (One-Period Rate Estimation Error):** Error committed during the estimation of individual one-period rates— $Y_2, Y_3, Y_4, \dots, Y_n$ .
2. **Type II Error (Single-Discount Rate Estimation Error):** Error committed during the estimation of a single discount rate as an estimator of the individual one-period rates ( $Y_1, Y_2, Y_3, \dots, Y_n$ ) either by using an averaging procedure (geometric, harmonic, arithmetic, or contraharmonic) or by using an estimation technique (OLS, GLS, 2SLS, 3SLS, or k-class estimator).

However, the use of one-period multiple rates to present value the future cash flows will result in only one type of error—Type I. If the Type II error is very small, then  $PV_s = PV_m$ . In general, however, due to the Type II error,  $PV_s$  will either overestimate or underestimate the true value of  $PV_m$ . The average value of the Type I errors may be zero. However, the average value of the effects of the Type-II error will not be equal to zero. In the present valuation models, therefore, the use of one-period multiple rates is superior to the use of a single discount rate.

The following numerical examples demonstrate that the Type II error is serious and nonvanishing.

Let:

- $Y_i^i$  = One-period rate expected to prevail during period  $i$ ; ( $i = 1, 2, 3, \dots, n$ )
- $h$  = Harmonic-mean of one-period rates ( $1+Y_i^i$ ); ( $i = 1, 2, 3, \dots, n$ )
- $g$  = Geometric-mean of one-period rates ( $1+Y_i^i$ ); ( $i = 1, 2, 3, \dots, n$ )
- $a$  = Arithmetic-mean of one-period rates ( $1+Y_i^i$ ); ( $i = 1, 2, 3, \dots, n$ )
- $PV^*$  = Present value using multiple single-period rates under the assumption that rates, on average, are consistently increasing over time ( $Y_1 < Y_2 < Y_3 < \dots < Y_n$ ).
- $PV^{**}$  = Present value using multiple single-period rates under the assumption that rates, on average, are consistently decreasing over time ( $Y_1 > Y_2 > Y_3 > \dots > Y_n$ ).
- $PV^{***}$  = Present value under the assumption that the rates do not change ( $Y_1 = Y_2 = Y_3 = \dots = Y_n$ ), or, if they do change, then they change in a neutralizing fashion.
- $PVH$  = Present value of future net cash flows using harmonic average of one-period rates ( $1+Y_i$ ;  $i = 1, 2, 3, \dots, n$ ).
- $PVG$  = Present value of future net cash flows using geometric average of the one-period rates ( $1+Y_i$ ;  $i = 1, 2, 3, \dots, n$ ).
- $PVA$  = Present value of future net cash flows using arithmetic average of one-period rates ( $1+Y_i$ ;  $i = 1, 2, 3, \dots, n$ ).

The three means of the one-period rates can be defined as:

Equation 1.1

$$h = N / [1/(1+Y_1^0) + 1/(1+Y_1^1) + 1/(1+Y_1^2) + 1/(1+Y_1^3) + \dots + 1/(1+Y_1^{n-1})]$$

Equation 1.2

$$g = [(1+Y_1^0)(1+Y_1^1)(1+Y_1^2)(1+Y_1^3)(1+Y_1^4) \dots (1+Y_1^{n-1})]^{(1/N)}$$

Equation 1.3

$$a = [(1+Y_1^0) + (1+Y_1^1) + (1+Y_1^2) + (1+Y_1^3) + \dots + (1+Y_1^{n-1})] / N$$

The present value of the project's future net cash flows using multiple one-period current and expected future rates,  $PV^*$ , can be defined as:

Equation 1.4

$$PV^* = d_1X_1 + d_2X_2 + d_3X_3 + \dots + d_nX_n$$

where:

Equation 1.5

$$d_j = \prod_{i=0}^{j-1} (1/(1+Y_1^i)); \quad (j = 1, 2, 3, \dots, n)$$

Present value of future net cash flows using the harmonic average of the multiple one-period current and expected future rates  $(1+Y_1^i)$ ,  $PVH$ , can be defined as:

Equation 1.6

$$PVH = h_1X_1 + h_2X_2 + h_3X_3 + \dots + h_nX_n$$

where:

Equation 1.7

$$h_j = 1 / \left[ \sum_{i=0}^{j-1} ((i) / (1+Y_1^i)) \right]; \quad (j = 1, 2, 3, \dots, n)$$

Present value of future net cash flows using the geometric average of the multiple one-period current and expected future rates  $(1+Y_1^i)$ ,  $PVG$ , can be defined as:

Equation 1.8

$$PVG = g_1X_1 + g_2X_2 + g_3X_3 + \dots + g_nX_n$$

where:

Equation 1.9

$$g_j = (1 / \left[ \prod_{i=0}^{j-1} (1+Y_1^i) \right]^{(1/j)}); \quad (j = 1, 2, 3, \dots, n)$$

Present value of future net cash flows using the arithmetic average of the multiple one-period current and expected future rates  $(1+Y_1^i)$ , PVA, can be defined as:

Equation 1.10

$$PVA = a_1X_1 + a_2X_2 + a_3X_3 + \dots + a_nX_n$$

where:

Equation 1.11

$$a_j = (1 / [\sum_{i=0}^{j-1} (1+Y_1^i)] / j); \quad (j = 1, 2, 3, \dots, n)$$

In the following examples, it is shown that if the rates have an increasing (decreasing) trend, then the PVH, PVH, and PVA will be smaller (larger) than PV\* (PV\*\*), indicating that any type of average (geometric, arithmetic, or harmonic) of rates used as a single discount rate will underestimate (overestimate) the present value of the future cash flows. And hence it would be more appropriate to use the multiple single-period rates as discounting factors, rather than a single rate (some average of the multiple single-period rates) to present-value the future cash flows. Only if the rates remain constant or if they change in a neutralizing fashion, the present value of the future net cash flows using any method will yield the identical results, that is, PVH = PVG = PVA = PV\*\*\*.

#### Case 1 (Increasing Trend In One-Period Rates):

$$Y_1^0 < Y_1^1 < Y_1^2$$

Let:  $N = 3$ ,  $Y_1^0 = .1$ ,  $Y_1^1 = .2$ ,  $Y_1^2 = .3$ , and  $X_i = 120$ ; ( $i = 1, 2, 3$ )

The values of the three means are:

$$\begin{aligned} h &= 0.19443155452 \\ g &= 0.19721576726 \\ a &= 0.20 \end{aligned}$$

The present value using one-period rates is:

$$PV^* = 269.93$$

The present value of the future cash flows using the harmonic average of  $1+Y_i$ , PVH, is given by:

$$PVH = 255.172086397$$

The present value of the future cash flows using the geometric average of  $1+Y_i$ , PVG, is given by:

$$PVG = 253.970087127$$

The present value of the future cash flows using the arithmetic average of  $1+Y_i$ , PVA, is given by:

$$PVA = 252.77$$

Since,  $h < g < a$

Therefore,  $PVH > PVG > PVA$

Furthermore, if the rates have a consistently increasing trend (i.e., first year rate is lower than the second year rate which, in turn, is lower than the third year rate, and so on) as was the case in the above example, then:

$$PV^* > PVH > PVG > PVA$$

The above relationship also holds true for the case where average positive deviations in one-period rates are greater towards the end of the investment horizon than towards the beginning of the investment horizon, as it would amount to an increasing market rate trend and the one-rate discounting method will yield a higher value for the present value of the future cash flows.

### Case 2 (Decreasing Trend In One-Period Rates):

$$Y_1^0 > Y_1^1 > Y_1^2$$

Let:  $Y_1^0 = .3, Y_1^1 = .2, Y_1^2 = .1, N = 3$  and  $X_i = 120; (i = 1, 2, 3)$

then the relationship among h, g, and a will not change (i.e.,  $h < g < a$  and hence  $PVH > PVG > PVA$ ), however, the present value using multiple single-period rates ( $PV^{**}$ ) will be the smallest:

$$PV^{**} = 239.1608391$$

and, therefore,

$$PVH > PVG > PVA > PV^{**}$$

The above relationship also holds true for the case where average positive deviations in one-period rates are greater in the beginning of the investment horizon than towards the end of the investment horizon, as it would amount to a declining interest rate trend and the one-rate discounting method will yield a lower value for the present value of the future cash flows.

### Case 3 (Constant In One-Period Rates):

$$Y_1^0 = Y_1^1 = Y_1^2$$

If the rates remain constant over the project's life (for example,  $Y_1^0 = Y_1^1 = Y_1^2 = .1$ ), then  $h = g = a$ , and the multiple single-period rate discounting method will neither underestimate nor overestimate the present value of future cash flows, that is:

$$PV^{***} = PVH = PVG = PVA$$

The above relationship also holds true for the case where the rates fall and rise but in a neutralizing fashion (that is they fall 50% of the time and they rise 50% of the time without any trend such that they neutralize each other and the average turns out to be equal to the first year rate).

In the above examples, the future cash flows were assumed to be constant (\$120) from period to period. The above results hold true even if the cash flows vary from period to period.

Thus to correctly estimate the net rate of return on a project, the use of one-period rates is superior to the use of a single discount rate obtained by taking an average of the one-period rates as suggested by the Expectations School of the term structure theory.

## THE OPPORTUNITY-COST-ADJUSTED RATE OF RETURN: AN ESTIMATOR OF INFLATION

### The Basic Idea And Intuitive Reasoning

Suppose an investor buys a 10% interest-paying bond. If all other economic conditions remain unchanged during the investment horizon, then the rate of return on this bond will be 10%. Imagine what will happen if the bond (the piece of paper) has some extra positive qualities such that the holder is suddenly transformed into a handsome young person with a divine look (blonde hair, blue eyes, 180 lbs., and 6'-5"). The person suddenly becomes attractive to all the individuals of the opposite persuasion. Suppose the bond has some other extra qualities: it brings stability in the marital relationship and eradicates germs and diseases in a 100 yard radius. If the investor's benefits derived through the bond's extra positive qualities can be measured and assigned a monetary value of 9% of the bond's price, then the real rate of return on the bond will be 19% instead of 10%.

Equally likely, the bond may have only negative extra qualities. The bondholder may become accident-prone (mafia notices the wealth) or neighbors may become jealous and begin to socially boycott the bondholder. The investor may also become schizophrenic due to potential asset-price fluctuations. If the loss arising due to the negative qualities of the bond can be measured and assigned a monetary value of -7% of the bond's price, then the real rate of return on the bond is only 3%. If the loss rate is -10%, then the real rate of return is 0%, while if the real loss rate exceeds 10%, say 17%, then the real rate of return is -7%.

Consider a one-year single-payment bond (or a project) with face value  $F$  and price  $P$ . The total interest received by an investor, then, is:  $F-P$ . The total interest,  $F-P$ , is also the total opportunity cost of the price paid for this bond. The opportunity cost may, among other things, include compensation for the loss of purchasing power (inflation), compensation for the expected loss of revenue due to default, and compensation for the loss of revenue due to other negative features of the bond or the project (callability, currency control, expropriation, taxability, maturity, transaction costs, etc.). At the time of the purchase of the bond or the project,  $F-P$  is the estimated total opportunity cost. However, as time progresses economic conditions may change such that the actual opportunity cost of the price paid for the bond may increase (or decrease). Thus the actual total opportunity cost of the investment asset may be higher (or lower) than  $F-P$ . However, the investor will receive only  $F-P$  as a compensation. *The sacrifice made by the investor to receive a total interest of  $F-P$  may no longer be just  $P$  but an amount higher (lower) than  $P$ .* After adjustment for the increase (decrease) in the opportunity cost, the actual price,  $P_a$ , paid for the bond will be higher (lower) than  $P$ .

As an example, suppose  $F=1000$ ,  $P=900$ , and the opportunity cost increases by \$50 during the investment horizon such that  $P_a = 900 + 50$ . Then the increased-opportunity-cost adjusted actual rate of return,  $R_a$ , on this bond will be:

$$R_a = \frac{[\text{Total Revenue} - \text{Total Cost}]}{\text{Total Cost}} = \frac{F - P_a}{P_a} = \frac{(1000-950)}{950} = 5.26\%$$

If during the investment horizon the opportunity cost rises by \$100, then:

$$R_a = (1000-1000)/1000 = 0$$

And if during the investment horizon, the opportunity-cost rises by an amount greater than \$100, say \$145, then:

$$R_a = (1000-1045)/1045 = -(4.31)\%$$

Calculating the rate of return,  $R$ , in the traditional fashion,

$$R = \frac{F - P}{P} = \frac{(1000-900)}{900} = 11.11\%.$$

Thus the increased-opportunity-cost adjusted actual rate of return is lower than the rate of return implied by the indenture of the bond. Note the fact that if the opportunity costs increase significantly, then the opportunity-cost-adjusted rate of return could be zero or negative. Therefore, under the changing opportunity cost conditions, the conventional argument that the rate of return on a single-payment bond that is held to its maturity is equal to  $[(F-P)/P]$  is misleading if not incorrect.  $R_a$ -R may be considered as an estimator for the increased cost conditions in general and inflation rate in particular. It is the opportunity-cost-adjusted rate of return that a client (investor) should be made aware of by financial managers.

In the subsequent analysis, the above discussion has been extended to n-period single-payment bonds/projects and n-period multiple revenue-generating bonds/projects. This extension entails another debate over the choice of an appropriate measure of mean (central tendency) -geometric-mean, harmonic-mean, and arithmetic-mean- to annualize the rate of return on n-period investment projects. The ensuing analysis indicates that all the three measure of central tendency have important roles to play and they are complementary to each other in the investment decisions process.

### Multi-Period Single-Payment Investment Assets

Consider the simple case of an n-period single-payment bond (or a project) with face value F and price P. The market rate or the yield to maturity on an n-period bond at time 0 is  $Y_n^0$ , while the opportunity-cost-adjusted one-period rate during time 0 is  $Y_{1a}^0$ . The opportunity-cost-adjusted one-period rate during period i is given by  $Y_{1a}^i$  (i = 1, 2, 3, ..., n-1). Based upon the traditional approach, the (unadjusted) rate of return,  $R_u$ , on this bond is given by:

Equation 2

$$R_u = [(F/P)^{(1/N)} - 1] = Y_n^0$$

where:  $P = F/(1+Y_n^0)^N$

If the bond is held to maturity then the rate of return on this bond, according to the literature, will equal its yield to maturity,  $Y_n^0$ . However, during the life of this bond, the rates will change numerous times, which will change the opportunity cost of the price paid. If the rates on the average increase (decrease), then the opportunity cost of the price of the bond will increase (decrease) and as a result the adjusted rate of return ( $R_a$ ) on this bond will be lower (higher) than the unadjusted rate of return, ( $R_u$ ). The opportunity cost-adjusted price,  $P_a$ , of the bond and the corresponding opportunity cost-adjusted rate of return on the bond,  $R_a$ , can be defined as:

Equation 3.1

$$P_a = P\{[1+(Y_{1a}^0 - Y_n^0)]\{1+(Y_{1a}^1 - Y_n^0)\}\{1+(Y_{1a}^2 - Y_n^0)\}\{1+(Y_{1a}^3 - Y_n^0)\}\{1+(Y_{1a}^{n-1} - Y_n^0)\}\}$$

or

Equation 3.2

$$P_a = P [(1+d_0)(1+d_1)(1+d_2)(1+d_3)...(1+d_{n-1})]$$

or ( $d_i$  is defined in equation 4)

Equation 3.3

$$P_a = P [1 + d]$$

where:  $[1 + d] = [(1+d_0)(1+d_1)(1+d_2)(1+d_3)...(1+d_{n-1})]$

Using equation 2 for P, equation 3.3 can be rewritten as:

Equation 3.4

$$P_a = [F/(1+Y_n^0)^N] [1 + d_g]^N$$

where:

Equation 4

$$d_i = (Y_{1a}^i - Y_n^0); (i = 0, 1, 2, 3, \dots, n-1) = \text{differential rate}$$

Equation 5

$$1 + d_g = [(1+d_0)(1+d_1)(1+d_2)(1+d_3)\dots(1+d_{n-1})]^{(1/N)} = \text{geometric mean of } 1+d_i; (i=1,2,3,\dots,n-1)$$

Assume that the absolute value of the differential rate,

Equation 6

$$|d_i = (Y_{1a}^i - Y_n^0)| < 1$$

(The above assumption is made due to the simple fact that even a single  $d_i$  that assumes a negative value smaller than one will yield a perverse result, the opportunity-cost-adjusted price,  $P_a$ , will be negative, for which no meaningful interpretation can be derived. The restriction can be relaxed by specifying an additive version of the above expression and by adopting a harmonic-mean-based approach or an arithmetic-mean-based approach.)

Equation 7.1

$$R_a = [(F/P_a)^{(1/N)} - 1$$

Using equation 2 for P in equation 3.1 and then using equation 3.1 for  $P_a$ , equation 7.1 can be rewritten as:

Equation 7.2

$$R_a = [(1+Y_n^0)^N / \{1+(Y_{1a}^0 - Y_n^0)\} \{1+(Y_{1a}^1 - Y_n^0)\} \{1+(Y_{1a}^2 - Y_n^0)\} \{1+(Y_{1a}^3 - Y_n^0)\} \dots \{1+(Y_{1a}^{n-1} - Y_n^0)\}]^{(1/N)} - 1$$

Using equation 4, equation 7.2 can be rewritten as:

Equation 7.3

$$R_a = (1+Y_n^0) / [(1+d_0)(1+d_1)(1+d_2)(1+d_3) \dots (1+d_{n-1})]^{(1/N)} - 1$$

Using equation 5, equation 7.3 can be rewritten as:

Equation 7.4

$$R_a = [(1+Y_n^0) / (1+d_g)] - 1$$

Based upon equation 7.4, the following conclusions can be derived:

$$\begin{aligned}
 R_a = 0 & \quad \text{if } d_g = Y_n^0 \\
 R_a < 0 & \quad \text{if } d_g > Y_n^0 \\
 R_a > 0 & \quad \text{if } d_g < Y_n^0
 \end{aligned}$$

The above results indicate that when the average of the differential rates equals the yield to maturity on the n-period bond, then the opportunity-cost-adjusted rate of return will be equal to zero. The opportunity cost of the price paid for the n-period bond has risen by an amount equal to the total interest received on the bond (i.e., F-P) such that the investor's net gain from holding the n-period investment instrument is zero. However, if the average of the differential rates is larger than the yield to maturity on the n-period bond, then the opportunity cost of the price paid is larger than the total interests (F-P) received, and thus, in net, the investor has lost. Similarly, if the average of the differential rates is less than the yield to maturity on an n-period bond, then the opportunity cost of the price paid is less than the total interests (F-P) received, and the investor, in net, has some positive return, even though not as high as originally intended (i.e.,  $Y_n^0$ ). The investor will receive his/her desired rate of return,  $Y_n^0$ , on the n-period bond only if the average of the differential rate is equal to zero. The average of the differential rate will be zero either when the yield curve is horizontal all the time without any shifts, or when increases and decreases in the market rates occur in a neutralizing fashion (the amount of increase is equal to the amount of decrease).

Thus the actual opportunity-cost-adjusted rate of return could be positive, negative, or zero depending upon the magnitude of the average of the differential rates. The one-period rates can be higher or lower than the n-period rate due to a number of factors (inflation, risk, and other features). The long-term instruments are considered riskier than the short-term instruments. Assuming the two types of instruments (n-period and one-period) are identical in every respect, the inflation rate can cause substantial deviation in their rates. The long-term rate will remain constant during the life of the instrument. But the short-term rates will increase (decrease) with a rise (fall) in inflation rate. Therefore, if the short-term rates, on the average, are higher than the long-term rates (i.e.,  $D_g > 0$ ), then it can be concluded that the purchasing power of the interest plus principal received on the n-period bond has decreased, resulting in a negative real rate of return. Similarly, if the short-term rates, on the average, are lower than the long-term rates (i.e.,  $D_g < 0$ ), then it can be concluded that the purchasing power of the interest plus principal received on the n-period bond has increased, resulting in a positive real rate of return. *THUS THE VALUE OF THE DIFFERENTIAL OF  $R_u$  AND  $R_b$  CAN BE CONSIDERED AS AN ESTIMATOR FOR THE AVERAGE UNEXPECTED CHANGES IN THE INFLATION RATE AND OTHER COSTS DURING THE LIFE OF THE N-PERIOD BOND (OR THE INVESTMENT HORIZON)*. Based upon equation 7.4, an additional set of conclusions can be derived:

$$\begin{aligned}
 \text{For } d_g = 0, R_a &= R_u \text{ and } P_a = P \\
 \text{For } d_g > 0, R_a &< R_u \text{ and } P_a > P \\
 \text{For } d_g < 0, R_a &> R_u \text{ and } P_a < P
 \end{aligned}$$

If on average the one-period rates are equal to the n-period rate (i.e., the average of these differential rates is zero), then the opportunity-cost-adjusted and unadjusted rates of return will be identical. However, if on average the one-period rates are higher than the n-period rate (i.e., the average of these differential rates is positive), then the opportunity-cost-adjusted rate of return will be lower than the unadjusted rate of return. Conversely, if on average the one-period rates are lower than the n-period rate (i.e., the average of these differential rates is negative), then the opportunity-cost-adjusted rate of return will be higher than the unadjusted rate of return.

Equation 7.4 and the subsequent conclusions are derived using the geometric mean of the differential rates. The above conclusions will hold true with respect to the harmonic mean or the arithmetic mean of the differential rates ( $d_h$  and  $d_a$ ) as well. Furthermore, the above analysis has been done for a single-payment bond, the same will hold true for the "present valuation" of a project that generates only one cash flow towards the end of the terminal time period.

A Numerical Example (2 Periods):

$$N = 2; F = 1,000; Y_2^0 = .12; Y_{1a}^0 = .10; Y_{1a}^1 = .25$$

$$P = 797.1939$$

$$P_a = 882.8125$$

$$R_u = [(F/P)^{(1/2)} - 1] = Y_2^0 = .12$$

$$R_a = [(F/P_a)^{(1/2)} - 1] = .0643$$

The opportunity cost-adjusted rate of return is only 6.43% which is below the originally desired rate of return of 12%.  $R_a$  will be negative, positive, or zero, depending upon the value of the average of the differential rates. If the average of this differential rate is 1.12, then the adjusted-rate of return will be zero and a value above (below) 1.12 will result in negative (positive) rate of return. If the second year one-period rate is 40%, then the geometric average of the differential rates will be equal to 1.12. If the second year one-period rate is above (below) 40%, then the geometric average of the differential rates will be greater (smaller) than 1.12. Thus if the second year one-period rate increases to 40%, then  $P_a = 1,000$  and  $R_a = 0$ . If the second year one-period rate increases to 43%, then the price of the bond will be above the face value and the rate of return will be negative ( $P_a = 1,023.4375$  and  $R_a = -.0115$ ). If the second year rate is 37%, then  $P_a = 976.56$  and  $R_a = .01192$ .

$$\text{For } d_g = 0, \quad R_a = R_u = .12 \text{ and } P_a = P = 797.12$$

$$\text{For } d_g = .05 > 0, \quad [P_a=837.05357] > [P=797.19387] \text{ and } [R_a=.066667] < [R_u = .12]$$

$$\text{For } d_g = -.07 < 0, \quad [P_a=757.3341] < [P=797.19387] \text{ and } [R_a=.2043] > [R_u = .12]$$

## THE MEAN-BASED RATES OF RETURN: A NEW APPROACH

### Multiple Revenue Generating Investment Assets/Projects

In this section, the analysis of Section I has been extended to apply to an n-period coupon paying bond and to an n-period revenue-generating project.

### Price Adjustment For Changing Opportunity Costs

Future value (FV<sub>i</sub>) of the project's ith period's net cash inflow (X<sub>i</sub>) can be defined as:

Equation 8

$$FV_i = X_i [(1+Y_1^i)(1+Y_1^{i+1})(1+Y_1^{i+2})(1+Y_1^{i+3}) \dots (1+Y_1^{n-1})]; \quad (i = 1, 2, 3, \dots, n)$$

The present value (PV<sub>i</sub>) of the ith period's net cash inflow (X<sub>i</sub>) can be defined as:

Equation 9

$$PV_i = \frac{X_i [(1+d_0)(1+d_1)(1+d_2)(1+d_3)\dots(1+d_{n-1})]}{(1+Y_1^0)(1+Y_1^1)(1+Y_1^2)(1+Y_1^3) \dots (1+Y_1^{i-1})} \quad (i = 1, 2, 3, \dots, n)$$

where  $d_i$  represents the differential rate between the opportunity-cost adjusted (actual) one-period market rate,  $Y_{1a}^i$ , and the unadjusted (expected) one-period market rate,  $Y_1^i$ , during period i, that is:

Equation 10

$$d_i = Y_{1a}^i - Y_1^i \quad (i = 1, 2, 3, \dots, n)$$

As in the previous case [see inequality (6)], assume that the absolute value of the differential rate is less than unity:

$$|d_i = (Y_{1a}^i - Y_1^i)| < 1$$

(The above assumption is made due to the simple fact that even a single  $d_i$  that assumes a negative value smaller than one will yield a perverse result, the opportunity-cost-adjusted price,  $P_a$ , will be negative, for which no meaningful interpretation can be derived. The restriction can be relaxed by specifying an additive version of the above expression and by adopting a harmonic-mean-based approach or an arithmetic-mean-based approach.)

The aggregate present value (PV) of the project's future net cash inflows ( $X_i$ ;  $i = 1, 2, 3, \dots, n$ ) is given by:

Equation 11

$$PV = PV_1 + PV_2 + PV_3 + PV_4 + \dots + PV_{n-3} + PV_{n-2} + PV_{n-1} + PV_n$$

Substituting the values of the individual  $PV_i$ s ( $i = 1, 2, 3, \dots, n$ ) from equation 9 into equation 11 yields:

Equation 12

$$\begin{aligned} PV = & [(1+Y_1^0)(1+Y_1^1)(1+Y_1^2)(1+Y_1^3)(1+Y_1^4) \dots (1+Y_1^{n-1})]^{-1} \\ & [(1+d_0)(1+d_1)(1+d_2)(1+d_3) \dots (1+d_{n-1})] \\ & \{X_1(1+Y_1^1)(1+Y_1^2)(1+Y_1^3)(1+Y_1^4) \dots (1+Y_1^{n-1}) \\ & + X_2(1+Y_1^2)(1+Y_1^3)(1+Y_1^4)(1+Y_1^5) \dots (1+Y_1^{n-1}) \\ & + X_3(1+Y_1^3)(1+Y_1^4)(1+Y_1^5)(1+Y_1^6) \dots (1+Y_1^{n-1}) \\ & + X_4(1+Y_1^4)(1+Y_1^5)(1+Y_1^6)(1+Y_1^7) \dots (1+Y_1^{n-1}) \\ & \cdot \\ & \cdot \\ & \cdot \\ & + X_{n-3}(1+Y_1^{n-3})(1+Y_1^{n-2})(1+Y_1^{n-1}) \\ & + X_{n-2}(1+Y_1^{n-2})(1+Y_1^{n-1}) \\ & + X_{n-1}(1+Y_1^{n-1}) \\ & + X_n\} \end{aligned}$$

The aggregate future value (FV) of the project's net cash inflows is given by:

Equation 13

$$FV = FV_1 + FV_2 + FV_3 + FV_4 + \dots + FV_{n-3} + FV_{n-2} + FV_{n-1} + FV_n$$

Substituting the values of individual  $FV_i$ s ( $i = 1, 2, 3, \dots, n$ ) from equation 8 into equation 10 yields:

Equation 14

$$\begin{aligned} FV = & [(X_1(1+Y_1^1)(1+Y_1^2)(1+Y_1^3)(1+Y_1^4) \dots (1+Y_1^{n-1}) \\ & + X_2(1+Y_1^2)(1+Y_1^3)(1+Y_1^4)(1+Y_1^5) \dots (1+Y_1^{n-1}) \\ & + X_3(1+Y_1^3)(1+Y_1^4)(1+Y_1^5)(1+Y_1^6) \dots (1+Y_1^{n-1}) \\ & + X_4(1+Y_1^4)(1+Y_1^5)(1+Y_1^6)(1+Y_1^7) \dots (1+Y_1^{n-1}) \\ & \cdot \\ & \cdot \end{aligned}$$

$$\begin{aligned}
& + X_{n-3}(1+Y_1^{n-3})(1+Y_1^{n-2})(1+Y_1^{n-1}) \\
& + X_{n-2}(1+Y_1^{n-2})(1+Y_1^{n-1}) \\
& + X_{n-1}(1+Y_1^{n-1}) \\
& + X_n]
\end{aligned}$$

The overall ratio (L) of the future value of the project's net cash inflows to the project's net present value is given by:

Equation 15

$$L = \frac{FV}{PV} = \frac{[(1+Y_1^0)(1+Y_1^1)(1+Y_1^2)(1+Y_1^3)(1+Y_1^4) \dots (1+Y_1^{n-1})]}{[(1+d_0)(1+d_1)(1+d_2)(1+d_3)\dots(1+d_{n-1})]}$$

Using the values of  $FV_i$  and  $PV_i$  from equations 8 and 9, their ratio ( $L_i$ ) can be defined as:

Equation 16

$$L_i = \frac{FV_i}{PV_i} = \frac{[(1+Y_1^0)(1+Y_1^1)(1+Y_1^2)(1+Y_1^3)(1+Y_1^4) \dots (1+Y_1^{n-1})]}{[(1+d_0)(1+d_1)(1+d_2)(1+d_3)\dots(1+d_{n-1})]} \quad (i = 1, 2, 3, \dots, n)$$

Equation 17

$$\frac{[(1+Y_1^0)(1+Y_1^1)(1+Y_1^2)(1+Y_1^3)(1+Y_1^4) \dots (1+Y_1^{n-1})]}{[(1+d_0)(1+d_1)(1+d_2)(1+d_3) \dots (1+d_{n-1})]} > < 1$$

$$\text{as: } [(1+d_0)+(1+d_1)+(1+d_2)+(1+d_3)+\dots+(1+d_{n-1})] > < 0$$

The comparison of equations 15 and 16 reveals that:

Equation 18

$$L = L_1 = L_2 = L_3 = \dots = L_n$$

The ratio of individual  $FV_i$  and  $PV_i$  is identical to the ratio of the (total) FV and PV. Furthermore, all the individual ratios are identical. This is not a counterintuitive result since  $FV_i$  is the terminal value of each net cash inflow and  $PV_i$  is the opportunity-cost-adjusted terminal value of each price segment. Therefore, the ratio (and thereby the rate of return) of the future value to present value of any amount ( $X_i$  or the sum of  $X_i$ s;  $i = 1, 2, 3, \dots, n$ ) will be identical. Note the fact that in equations 15 and 16, the ratios ( $L = L_i$ ;  $i = 1, 2, 3, \dots, n$ ) turn out to be a ratio between the product of the one-period market rates  $[(1+Y_1^i)$ ;  $i = 1, 2, 3, \dots, n-1$ ] and the product of the differential rates  $[(1+d_i)$ ;  $i=1, 2, 3, \dots, n-1$ ] that prevail during the investment horizon (i.e., between period 0 and n).

The statement in inequality (17) reveals that if on the average the differential rates are positive (negative), then the ratio,  $L = L_i$ , will be smaller (larger) than the product of the individual one-period expected market rates  $[(1+Y_i)$ ;  $i = 1, 2, 3, \dots, n$ ]. However, if on the average the differential rates are zero, then the ratio of FV to PV will simply be the product of the individual one-year expected market rates  $[(1+Y_i)$ ;  $i = 1, 2, 3, \dots, n$ ]. For this

(i.e.,  $L_i = L = 1$ ), each opportunity-cost adjusted one-period market rate need not be equal to the corresponding one-period market rate. The ratio  $L_i=L$  can be equal to unity even if the opportunity-cost-adjusted rates are higher and lower than their corresponding one-period expected market rates during the investment horizon provided that there deviation is normally distributed (i.e., the rates are up and down in a neutralizing fashion).

$FV_i$  is the future value of the  $i$ th cash flow at the end of period  $N$  and  $PV_i$  is the opportunity cost-adjusted price for the  $i$ th cash flow also evaluated at the end of period  $N$  ( $i = 1, 2, 3, \dots, n$ ). To obtain, a mean-based (geometric, harmonic, or arithmetic) overall annual rate of return on a project, two steps are required. First, an annualized rate of return pertaining to the  $i$ th cash flow is to be obtained. Since the total profit of the  $i$ th period,  $FV_i - PV_i$  ( $i = 1, 2, 3, \dots, n$ ), is a lump sum amount accumulated in  $N$  periods, it has to be redefined (decomposed) into annual profits in order to obtain an annualized rate of return on the cash flow of the  $i$ th period. Second, an overall rate of return on the project is to be obtained by taking an average of all the individual rates of return pertaining to each net cash inflow. The overall rate of return on the project can be obtained either by adopting the same type of averaging procedure (geometric, harmonic, or arithmetic) in both the steps, or by adopting a mixture of the averaging procedures. The mixed averaging procedure can be defined as a procedure in which two different means are used in the two steps. For example, if a geometric averaging procedure is used in the calculation of the annualized rate of return on the individual net cash inflows (first step), then a non-geometric (harmonic or arithmetic) averaging procedure is used in the calculation of the overall rate of return on the project (second step) and vice versa. The mixed averaging procedure will be used in the next section. For this section, it is assumed that only one type of averaging procedure is used to calculate the annualized rate of return on the  $i$ th net cash inflow as well as the annualized rate of return on the project. The rate of return on an  $n$ -period project through geometric-, harmonic-, and arithmetic-mean-based approaches are defined in the following subsections.

### Geometric-Mean-Based Rate Of Return

It costs  $PV_i$  to receive  $FV_i$  at the end of the terminal time period  $N$ . The geometric-mean-based rate of return corresponding to the  $i$ th net cash inflow,  $R_{gi}$ , can thus be defined as:

Equation 19

$$R_{gi} = [FV_i/PV_i]^{(1/N)} - 1 \quad (i = 1, 2, 3, \dots, n)$$

Substitution for the value of  $FV_i/PV_i$  from equations 16 yields:

Equation 20.i

$$R_{gi} = \frac{[(1+Y_1^0)(1+Y_1^1)(1+Y_1^2)(1+Y_1^3)(1+Y_1^4) \dots (1+Y_1^{n-1})]^{(1/N)}}{[(1+d_0)(1+d_1)(1+d_2)(1+d_3)\dots(1+d_{n-1})]^{(1/N)}} - 1 \quad (i = 1, 2, 3, \dots, n)$$

Since:

Equation 21

$$[FV_1/PV_1] = [FV_2/PV_2] = [FV_3/PV_3] = \dots = [FV_n/PV_n]$$

Therefore,

Equation 22

$$R_{g1} = R_{g2} = R_{g3} = \dots = R_{gi} = \dots = R_{gn}$$

as is evidenced through the expressions on the right hand sides of equations 20.1-20.n below:

Equation 20.1

$$R_{g1} = \frac{[(1+Y_1^0)(1+Y_1^1)(1+Y_1^2)(1+Y_1^3)(1+Y_1^4) \dots (1+Y_1^{n-1})]^{(1/N)}}{[(1+d_0)(1+d_1)(1+d_2)(1+d_3)\dots(1+d_{n-1})]^{(1/N)}} - 1$$

Equation 20.2

$$R_{g2} = \frac{[(1+Y_1^0)(1+Y_1^1)(1+Y_1^2)(1+Y_1^3)(1+Y_1^4) \dots (1+Y_1^{n-1})]^{(1/N)}}{[(1+d_0)(1+d_1)(1+d_2)(1+d_3)\dots(1+d_{n-1})]^{(1/N)}} - 1$$

Equation 20.3

$$R_{g3} = \frac{[(1+Y_1^0)(1+Y_1^1)(1+Y_1^2)(1+Y_1^3)(1+Y_1^4) \dots (1+Y_1^{n-1})]^{(1/N)}}{[(1+d_0)(1+d_1)(1+d_2)(1+d_3)\dots(1+d_{n-1})]^{(1/N)}} - 1$$

Equation 20.n

$$R_{gn} = \frac{[(1+Y_1^0)(1+Y_1^1)(1+Y_1^2)(1+Y_1^3)(1+Y_1^4) \dots (1+Y_1^{n-1})]^{(1/N)}}{[(1+d_0)(1+d_1)(1+d_2)(1+d_3)\dots(1+d_{n-1})]^{(1/N)}} - 1$$

The overall geometric-mean-based rate of return on the project,  $R_g$ , can be defined as:

Equation 23.1

$$R_g = [(FV_1/PV_1)(FV_2/PV_2)(FV_3/PV_3) \dots (FV_n/PV_n)]^{(1/N)} - 1$$

Alternatively, using equation 19, equation 23.1 can be rewritten as:

Equation 23.2

$$R_g = [(1+R_{g1})(1+R_{g2})(1+R_{g3}) \dots (1+R_{gn})]^{(1/N)} - 1$$

Thus the overall rate of return on the project,  $R_g$ , is a geometric mean of the geometric mean due to the fact that the individual  $R_{gi}$ s are geometric means in themselves.

Since by equation 22,

$$R_{g1} = R_{g2} = R_{g3} = \dots R_{gi} = \dots = R_{gn}$$

Therefore equation 23.2 can be rewritten as:

Equation 24

$$R_g = [(1+R_{gi})^N]^{(1/N)} - 1 = 1 + R_{gi} - 1 = R_{gi}$$

Using the value of  $R_{gi}$  from equation 20.i in equation 24 yields:

Equation 25

$$R_g = \frac{[(1+Y_1^0)(1+Y_1^1)(1+Y_1^2)(1+Y_1^3)(1+Y_1^4) \dots (1+Y_1^{n-1})]^{(1/N)}}{[(1+d_0)(1+d_1)(1+d_2)(1+d_3)\dots(1+d_{n-1})]^{(1/N)}} - 1$$

THUS THE GEOMETRIC-MEAN-BASED RATE OF RETURN ON THE SUM OF THE INDIVIDUAL CASH INFLOWS ( $X_1+X_2+X_3 + \dots + X_n$ ) IS THE SAME AS THE GEOMETRIC-MEAN-BASED RATE OF RETURN ON INDIVIDUAL NET CASH INFLOWS ( $X_i; i=1,2,3,\dots,n$ ).

Note the fact that the result in equation 25 can also be obtained by taking the Nth root of the overall FV/PV ratio defined in equation 15.

### Harmonic-Mean-Based Based Rate Of Return

The overall harmonic-mean-based rate of return,  $R_h$ , on the project can be defined as:

Equation 26

$$R_h = [N / ((PV_1/FV_1) + (PV_2/FV_2) + (PV_3/FV_3) + (PV_4/FV_4) + \dots + (PV_n/FV_n))] - 1$$

Utilizing equation 16 the harmonic-mean-based rate of return corresponding to the net cash inflow of the ith period,  $R_{hi}$ , can be defined as:

Equation 27

$$R_{hi} = \frac{(1/(1+d_0) + 1/(1+d_1) + 1/(1+d_2) + 1/(1+d_3) + \dots + 1/(1+d_{n-1}))}{(1/(1+Y_1^0) + 1/(1+Y_1^1) + 1/(1+Y_1^2) + 1/(1+Y_1^3) + 1/(1+Y_1^{n-1}))} - 1 \quad (i = 1, 2, 3, \dots, n)$$

By equation 18 or 22,

$$[FV_1/PV_1] = [FV_2/PV_2] = [FV_3/PV_3] = \dots = [FV_n/PV_n]$$

Therefore,

Equation 28

$$R_{h1} = R_{h2} = R_{h3} = \dots = R_{hi} = \dots = R_{hn}$$

The overall harmonic-mean-based rate of return on the project,  $R_h$ , can be defined as:

Equation 29.1

$$R_h = [N / (1/(1+R_{h1}) + 1/(1+R_{h2}) + 1/(1+R_{h3}) + 1/(1+R_{h4}) + \dots + 1/(1+R_{hn}))] - 1$$

Note the fact that  $R_h$  is a harmonic mean of the harmonic mean due to the fact that the individual  $R_{hi}$ s are harmonic means in themselves.

By equation 28

$$R_{h1} = R_{h2} = R_{h3} = \dots = R_{hi} = \dots = R_{hn}$$

Therefore the overall harmonic-mean-based rate of return from the project can be expressed as:

Equation 29.2

$$R_h = [N / (N/(1+R_{hi}))] - 1 = (1 + R_{hi}) - 1 = R_{hi}$$

Using the value of  $R_{hi}$  from equation 27.2 in equation 29.2 yields:

Equation 30

$$R_h = \frac{(1/(1+d_0) + 1/(1+d_1) + 1/(1+d_2) + 1/(1+d_3) + \dots + 1/(1+d_{n-1}))}{(1/(1+Y_1^0) + 1/(1+Y_1^1) + 1/(1+Y_1^2) + 1/(1+Y_1^3) + 1/(1+Y_1^{n-1}))} - 1$$

*THE HARMONIC-MEAN-BASED RATE OF RETURN ON THE SUM OF THE INDIVIDUAL CASH INFLOWS ( $X_1+X_2+X_3+\dots+X_n$ ) IS THE SAME AS THE HARMONIC-MEAN-BASED RATE OF RETURN ON INDIVIDUAL NET CASH INFLOWS ( $X_i$ ;  $i=1,2,3,\dots,n$ ).*

### Arithmetic-Mean-Based Rate Of Return

The arithmetic-mean-based rate of return for the project's net cash inflow of the  $i$ th period,  $R_{ai}$ , can be defined as:

Equation 31

$$R_{ai} = [(FV_i/PV_i)/N] - 1$$

Making use of the result in equation 16, equation 31 can be redefined as:

Equation 32

$$R_{ai} = \frac{[(1+Y_1^0)+(1+Y_1^1)+(1+Y_1^2)+(1+Y_1^3)+(1+Y_1^4)+\dots+(1+Y_1^{n-1})]/N}{[(1+d_0)+(1+d_1)+(1+d_2)+(1+d_3)+\dots+(1+d_{n-1})]/N} - 1 \quad (i = 1, 2, 3, \dots, n)$$

By equation 18 or 22,

$$[FV_1/PV_1] = [FV_2/PV_2] = [FV_3/PV_3] = \dots = [FV_n/PV_n]$$

Therefore,

Equation 33

$$R_{a1} = R_{a2} = R_{a3} = \dots = R_{ai} = \dots = R_{an}$$

The overall harmonic-mean-based rate of return on the project,  $R_a$ , can be defined as:

Equation 34.1

$$R_a = [(FV_1/PV_1) + (FV_2/PV_2) + (FV_3/PV_3) \dots (FV_n/PV_n)]/N - 1$$

Using equation 31 in equation 34.1 yields:

Equation 34.2

$$R_a = [(1+R_{a1}) + (1+R_{a2}) + (1+R_{a3}) + \dots + (1+R_{an})]/N - 1$$

Note the fact that  $R_a$  is an arithmetic mean of the arithmetic mean due to the fact that the individual  $R_{ai}$ s are arithmetic means in themselves. Using the relation of equation 33 in equation 34.2 yields:

Equation 35

$$R_a = [N (1+R_{ai})]/N - 1 = (1 + R_{ai}) - 1 = R_{ai}$$

Using the expression for  $R_{ai}$  from equation 32 in equation 35 yields:

Equation 36

$$R_a = \frac{[(1+Y_1^0)+(1+Y_1^1)+(1+Y_1^2)+(1+Y_1^3)+(1+Y_1^4)+ \dots +(1+Y_1^{n-1})]/N}{[(1+d_0)+(1+d_1)+(1+d_2)+(1+d_3)+\dots+(1+d_{n-1})]/N} - 1$$

*THE ARITHMETIC-MEAN-BASED RATE OF RETURN ON THE SUM OF THE INDIVIDUAL CASH INFLOWS ( $X_1+X_2+X_3+\dots+X_n$ ) IS THE SAME AS THE ARITHMETIC-MEAN-BASED RATE OF RETURN ON INDIVIDUAL NET CASH INFLOWS ( $X_i$ ;  $i=1,2,3,\dots,n$ ).*

## ADJUSTED RATES OF RETURN VERSUS NON-ADJUSTED RATES OF RETURN

The unadjusted rates of return will be equal to their corresponding adjusted-rates of return under the three mean-based procedures if  $d_i = 0$ . Therefore, the three mean-based unadjusted rates of return can be obtained by substituting the value of  $d_i = 0$  in equations 25, 30, and 36. If the opportunity costs of the project do not change or they change in a neutralizing fashion (the amounts of increase and decrease in the opportunity costs are equal) during the project's life time (that is,  $d_i = 0$ ;  $i=1,2,3,\dots,n-1$ ), then the overall rates of return on the project under the three mean-based procedures [c.f. equations 25, 30, and 36] can be defined as:

Equation 25'

$$R_g^0 \Big|_{d_i=0} = [(1+Y_1^0)(1+Y_1^1)(1+Y_1^2)(1+Y_1^3)(1+Y_1^4) \dots (1+Y_1^{n-1})]^{(1/N)} - 1$$

Equation 30'

$$R_h^0 \Big|_{d_i=0} = N / [1/(1+Y_1^0)+1/(1+Y_1^1)+1/(1+Y_1^2)+1/(1+Y_1^3)+ \dots +1/(1+Y_1^{n-1})] - 1$$

Equation 36'

$$R_a^0 \Big|_{d_i=0} = (1/N)[(1+Y_1^0)+(1+Y_1^1)+(1+Y_1^2)+(1+Y_1^3)+(1+Y_1^4)+ \dots +(1+Y_1^{n-1})] - 1$$

Equation 37

$$R_{gu} = R_{g0}$$

Equation 38

$$R_{hu} = R_{h0}$$

Equation 39

$$R_{au} = R_{a0}$$

where  $R_{g0}$ ,  $R_{h0}$ , and  $R_{a0}$  represent respectively, the geometric-mean-based, harmonic-mean-based, and arithmetic-mean-based rates of return when  $d_i = 0$  ( $i=1,2,3,\dots,n-1$ ) and  $R_{gu}$ ,  $R_{hu}$ , and  $R_{au}$  represent respectively, the geometric-mean-based, harmonic-mean-based, and arithmetic-mean-based rates of return when the adjustments for the changes in opportunity costs are not made.

Comparing the values of the three mean-based rates of return under variable opportunity costs [equations 25, 30, and 36] with those under constant opportunity costs [equations 25', 30', and 36'], the following conclusions can be drawn:

Equation 40

$$R_g = R_g^u = R_g^0 \quad ; \text{ as: } [(1+d_0)(1+d_1)(1+d_2)(1+d_3)\dots(1+d_{n-1})] = 0$$

Similarly,

Equation 41

$$R_h = R_h^u = R_h^0 \quad ; \text{ as: } [1/(1+d_0) + 1/(1+d_1) + 1/(1+d_2) + 1/(1+d_3) \dots + 1/(1+d_{n-1})] = 0$$

Finally,

Equation 42

$$R_a = R_a^u = R_a^0 \quad ; \text{ as: } [(1+d_0)(1+d_1)(1+d_2)(1+d_3)\dots(1+d_{n-1})] = 0$$

The general conclusion is that if on the average, the opportunity costs rise (fall), then the opportunity-cost adjusted rates of return will be lower (higher) than the unadjusted rates of return irrespective of the type of averaging (geometric, harmonic, or arithmetic) procedure is used. If the opportunity costs remain constant or change in a neutralizing fashion such that the average of their change is zero, then the adjusted and unadjusted rates of return will be identical. It is to be noted that the two procedures will yield identical results even if the opportunity cost is changing but it is changing in a neutralizing fashion. Generally, traditional models use a single discount rate in the "present valuation" of a project's net cash inflows and in the computation of the implied rate of return. Furthermore, they generally use geometric-averaging procedure. If the unadjusted rate of return of this paper are interpreted as the traditional model's rate of return, then the results in equations 40, 41, and 42 suggest that rates of return, computed under traditional models' framework, are more likely to be fraught with errors under changing market conditions. In particular, the rates of return computed under the traditional model's framework will tend to be inflated (deflated) if the opportunity-costs of investment, on the average, rise (fall).

**RISK PREFERENCE AND THE CHOICE OF A  
RATE OF RETURN CALCULATIVE TECHNIQUE  
(A Numerical Example)**

The purpose of this section is to show that all the three mean-based procedures produce equally appealing results and each one of them can be effective in an appropriate context. Consider the following market conditions:

$$\begin{aligned} N = 4, \quad Y_1^0 &= .1, \quad Y_1^1 = .2, \quad Y_1^2 = .3, \quad Y_1^3 = .4 \\ Y_{1a}^0 &= .11, \quad Y_{1a}^1 = .22, \quad Y_{1a}^2 = .33, \quad Y_{1a}^3 = .44 \\ d_0 &= .01, \quad d_1 = .02, \quad d_2 = .03, \quad d_3 = .04 \end{aligned}$$

Substituting the above values of the one-period expected rates and the differential rates in equations 25, 30, and 36, yield the following values of the three mean-based rates of return:

$$\begin{aligned} R_g &= 0.21468397616 \\ R_h &= 0.20984969852 \\ R_a &= 0.21951219512 \end{aligned}$$

If the opportunity costs remain constant (i.e.,  $d_0 = d_1 = d_2 = d_3 = 0$ ), then using equations 25', 30', and 36', the rates of return are:

$$\begin{aligned} R_g^u &= R_g^0 = 0.24497700445 \\ R_h^u &= R_h^0 = 0.2399483871 \\ R_a^u &= R_a^0 = 0.25 \end{aligned}$$

The above example demonstrates that:

$$\begin{aligned} R_h &< R_g < R_a \\ R_h^u &< R_g^u < R_a^u \end{aligned}$$

A harmonic-mean-based rate of return will always be lower than a geometric-mean-based rate of return which, in turn, will always be lower than an arithmetic-mean-based rate of return. The differences in the values of these three mean-based rates of return will be very small. The choice of a particular mean-based (harmonic, geometric, or arithmetic) technique to calculate the rate of return can be linked to the risk preference of the investors. With regard to the adoption of an investment project or to the purchase of an asset, therefore, a risk-averse investor would be more likely to make a decision based upon a harmonic-mean based rate of return, a risk preferrer investor would be more likely to make a decision based upon an arithmetic-mean-based rate of return, and a risk-neutral investor would be more likely to make a decision based upon a geometric-mean-based rate of return. The rate of return could be calculated through the use of any of the three measures of mean (central tendency): harmonic mean, geometric mean, and arithmetic mean. All the three measures provide competing results. Therefore, there is no strong reason to support the use of the geometric-mean approach as the superior approach to the harmonic-mean approach or to the arithmetic-mean approach as is generally the case in the existing literature.

**VARIOUS MEAN-BASED-RATES OF RETURN AS SUBSETS OF EACH OTHER**

In this section, a mixed averaging procedure will be adopted to show that one type of mean-based rate of return can be obtained as a special case of the other two. The mixed averaging procedure can be defined as a procedure in which two different means are used to obtain an overall rate of return on the project. If a geometric averaging procedure is used in the calculation of the annualized rate of return on the individual cash inflows (first step), then

a non-geometric (harmonic or arithmetic) averaging procedure is used in the calculation of the overall rate of return on the project (second step). In the following analysis, it will be shown that the procedures to calculate the geometric-mean-based rate of return and the arithmetic-mean-based rate of return are special cases of the procedure to calculate the harmonic-mean-based rate of return.

In Section II.3 [equation 26], the harmonic-mean-based rate of return was defined as:

Equation 26

$$R_h = [N / ((PV_1/FV_1) + (PV_2/FV_2) + (PV_3/FV_3) + (PV_4/FV_4) + \dots + (PV_n/FV_n))] - 1$$

Recall  $FV_i$  is the future value of the  $i$ th cash flow at the end of period  $N$  and  $PV_i$  is the opportunity cost-adjusted price for the  $i$ th cash flow also evaluated at the end of period  $N$  ( $i=1,2,3,\dots,n$ ). To obtain the overall rate of return on the project, derive a geometric-mean-based rate of return for each period's net cash inflows ( $X_i$ ;  $i=1, 2, 3, \dots, n$ ) and then take a harmonic average of all these individual rates. In Section II.2 [equation 20.i], the individual geometric-mean-based rate of return for the net cash inflows of the  $i$ th period,  $R_{gi}$ , was defined as:

Equation 20.i

$$R_{gi} = \frac{[(1+Y_1^0)(1+Y_1^1)(1+Y_1^2)(1+Y_1^3)(1+Y_1^4) \dots (1+Y_1^{n-1})]^{(1/N)}}{[(1+d_0)(1+d_1)(1+d_2)(1+d_3)\dots(1+d_{n-1})]^{(1/N)}} - 1 \quad (i = 1, 2, 3, \dots, n)$$

The overall harmonic-mean-based rate of return on the project,  $R_a$ , can be defined as:

Equation 43

$$R_h = [N / (1/(1+R_{g1}) + 1/(1+R_{g2}) + 1/(1+R_{g3}) + 1/(1+R_{g4}) + \dots + 1/(1+R_{gn-1}))] - 1$$

By equation 22,  $R_{g1} = R_{g2} = R_{g3} = \dots = R_{gi} = \dots = R_{gn}$ , therefore,

Equation 44

$$R_h = [N / (N(1/(1+R_{gi})))] - 1 = (1 + R_{gi}) - 1 = R_{gi}$$

Since the results of Section 3.2 [equation 24] indicate that:  $R_{gi} = R_g$ , therefore,

Equation 45

$$R_h = R_g$$

Under the mixed averaging procedure, the harmonic-mean of the individual geometric-mean-based rates of return is simply the geometric-mean based rate of return. If  $R_{gi}$ s (geometric-mean based rate of return on individual net cash inflows) are replaced by  $R_{ai}$  (the arithmetic-mean-based rate of return on individual net cash inflows) in equation (43), then:

Equation 46

$$R_h = R_a \quad [\text{Recall that: } R_{a1} = R_{a2} = R_{a3} = \dots = R_{ai} = \dots = R_{an} \text{ by equation 22.}]$$

That is, the harmonic-mean-based rate of return will simply be the arithmetic-mean-based rate of return. Thus the geometric-mean-based rate of return and arithmetic-mean-based rate of return are special cases of the harmonic-mean-based rate of return. Similarly, it can be shown that under the mixed averaging procedure, harmonic-mean-based rate of return and arithmetic-mean-based rate of return are special cases of the geometric-mean-based rate of return. Further, the harmonic-mean-based rate of return and the geometric-mean-based rate of

return can be shown to be the special cases of the arithmetic-mean-based rate of return. This is not a counterintuitive result, since each net cash inflow is evaluated at the end of the terminal time period, under the same market conditions, the rate of return on the overall project will be the same as the rate of return on the individual net cash inflow ( $X_i$ ) ( $i=1,2,3,\dots,n$ ). The rate of return on the sum of the individual cash inflows ( $X_1+X_2+X_3 +\dots+ X_n$ ) is the rate of return on an individual cash inflows ( $X_i$ ;  $i=1,2,3,\dots,n$ ). Therefore, the type of the mean-based procedure that is used to calculate the rate of return for the individual cash inflows (first step) will turn out to be the type of mean-based procedure for the calculation of the overall rate of return on the project in the final analysis (second step). The specification of a particular type of averaging procedure in the calculation of the rate of return on individual cash inflows, de facto, ties up the averaging procedure for the calculation of project's overall rate of return and hence the second step yields the same result as the first step.

## CONCLUSIONS

Traditionally, the calculation of the rate of return on a bond issue has been based upon the estimated opportunity costs of the investors. During the investment horizon, the opportunity costs, however, may change due to changing economic conditions. Therefore, the actual rate of return may be higher or lower, depending upon the direction of change, than the rate of return implied by the indenture of the bond. The rate of return could be calculated through the use of any of three measures of central tendency (means): harmonic mean, geometric mean, and arithmetic mean. All the three measures to calculate the rate of return provide equally appealing results. In contrast to conventional opinion, there is no strong evidence to support the view that the geometric-mean approach is superior to the harmonic-mean approach or to the arithmetic-mean approach. In fact, any one of the mean-based rates of return could be derived as a special case of the other two mean-based rates of return. Since the values of the rates of return based on harmonic-mean, geometric-mean, and arithmetic-mean techniques always lie in ascending order under variable rate conditions, the cautious risk-averse investor is more likely to choose the harmonic-mean-based rate of return evaluation technique, the risk-neutral investor is more likely to choose the geometric-mean-based rate of return evaluation technique, while the ambitious risk-preferrer investor is more likely to choose an arithmetic-mean-based rate of return evaluation technique. The use of a single discount rate (estimated either by any of the measures of central tendency or by econometric methods) would not be appropriate, as it would either overestimate or underestimate the present value of the future cash flows under variable market rate conditions. It would be more appropriate to use the multiple single-period rates as discounting factors, rather than a single rate (some average of the multiple single-period rates) to present-value the future cash flows. Only if the rates remain constant or if they change in a neutralizing fashion, the present value of the future net cash flows using any method will yield identical results. Since this paper's procedure to calculate the rate of return is different from the traditional model, any of the three measures (harmonic-mean, geometric-mean, and arithmetic-mean) can be used as a proxy for a single discount rate if so desired. However, the use of multi-period rates as the discounting factors in the capital budgeting models would still be superior.

**REFERENCES**

- [1] Bernhard, R.H., "Risk-Adjusted Values, Timing of Uncertainty Resolutions, and the Measurement of Project Worth," *Journal of Financial and Quantitative Analysis*, March 1984.
- [2] Capettini, R., R. Grimlund, and H. Toole, "Comment: The Unique Real Internal Rate of Returns," *Journal of Financial and Quantitative Analysis*, December 1979.
- [3] Clark, J.J., T. Hindelang, and R.E. Pritchard, *Capital Budgeting: Planning and Control of Capital Expenditures*, Prentice-Hall: Englewood Cliffs, New Jersey, 1989.
- [4] Copeland, T.E. and J.F. Weston, *Financial Theory and Corporate Policy*, Addison-Wesley Publishing: Reading, Massachusetts, 1988.
- [5] Cox, J.C., J.E. Ingersoll, and S.A. Ross, "A Reexamination of Traditional Hypothesis about the Term Structure of Interest Rates," *Journal of Finance* 36, September 1981, pp. 769-799.
- [6] Cox, J.C., J.E. Ingersoll, and S.A. Ross, "A Theory of the Term Structure of Interest Rates," *Econometrica* 53, March 1985, pp. 385-407.
- [7] Dorfman, R., "The Meaning of the Internal Rate of Return," *Journal of Finance* 36, December 1981, pp. 1010-1023.
- [8] Ho, Thomas S.Y. and Sang Bin Lee, "Term Structure Movements and Pricing Interest Rate Contingent Claims," *Journal of Finance* 41(5), 1986, pp. 1011-1030.
- [9] Hodder, J.E. and H.E. Riggs, "Pitfalls in Evaluating Risky Projects," *Business Review* 63, January-February 1985, pp. 128-135.
- [10] Jean, W.H., "Terminal Value or Present Value in Capital Budgeting Programs," *Journal of Financial and Quantitative Analysis* 6, January 1971, pp. 649-652.
- [11] Luckett, P.F., "ARR and IRR: A Review and an Analysis," *Journal of Business, Finance, and Accounting*, Spring 1984.
- [12] Oblak, D.J. and R.J. Helm, Jr., "Survey and Analysis of Capital Budgeting Methods Used by Multinationals," *Financial Management*, Winter 1981, pp. 37-41.
- [13] Reilly, Frank K., *Investment Analysis and Portfolio Management*, Dryden Press: Chicago, 1989.
- [14] Sick, G.A., "A Certainty Equivalence Approach to Capital Budgeting," *Financial Management* 15, Winter 1986, pp. 23-32.
- [15] Snow, Karl N., "Diagnosing Asset Pricing Models Using the Distribution of Asset Returns," *Journal of Finance*, July 1991, pp. 955-984.